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HYPERBOLIC STRUCTURE PRESERVING ISOMORPHISMS OF MARKOV SHIFTS

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ABSTRACT

In this paper we consider metric isomorphisms of Markov shifts which are also isomorphisms of the hyperbolic structures of the shift spaces. We prove that such isomorphisms need not be finitary, and that finitary isomorphisms need not preserve the hyperbolic structures unless they have finite expected code lengths. In particular we show that certain explicitly computable invariants previously associated with finitary isomorphisms with finite expected code lengths are, in fact, invariants of the hyperbolic structure of the Markov shifts.

1. Introduction

While the interplay between the metric and sequential (and hence topological) structures of Markov shifts has been investigated in great detail and has led to a number of significant results on the classification of Markov shifts under various notions of equivalence, ranging from measure preserving topological conjugacy to finitary isomorphism, very little is known about the connection between the metric and the hyperbolic structures of Markov shifts, or, equivalently, about the classification of Markov shifts under metric isomorphisms which are also isomorphisms (up to null sets) of the hyperbolic structures of the shift spaces. Such isomorphisms are not only of geometric interest, but they also appear to be linked intimately with the problem of classifying certain operator algebras associated with the Markov shift, and some of the invariants we obtain are related to dimension groups and modules associated with these algebras (cf. [3], [5], [6] and [15] for further details).

In this paper we construct a family of invariants of the hyperbolic structure of a Markov shift which allow us to conclude that metric isomorphisms of Markov shifts cannot — in general — be replaced by hyperbolic structure preserving isomorphisms in the sense decribed above. These invariants consist of two easily

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computable countable subgroups Γ and Δ of the multiplicative group of positive real numbers such that $\Gamma \supset \Delta$ and Γ/Δ is cyclic, a distinguished generator $c\Delta$ of the quotient group Γ/Δ, and the cohomology class of the information cocycle of Markov shift. Exactly the same invariants have arisen in earlier investigations by W. Krieger, W. Parry and the author in the context of finitary isomorphisms of Markov shifts with finite expected code (and inverse code) lengths (cf. [7], [9], and [10], where these invariants are discussed in some detail). This connection between hyperbolic structure preserving isomorphisms and those with finite expected code lengths is quite natural — at least in one direction: every isomorphism with finite expected code lengths is easily seen to induce an isomorphism of the hyperbolic structures of the Markov shifts (cf. Proposition 4.4). The remarkable feature of the results presented in this paper is that these invariants extend to a class of isomorphisms which need not even be finitary (cf. Theorem 4.3).

The paper is organized as follows: in §2 we consider an ergodic Markov shift T_P on a shift space (X_P, m_P), describe the hyperbolic structure of X_P, and introduce two equivalence relations, R_P and R'_P, on X_P which are associated with this hyperbolic structure. The group invariants Γ_P and Δ_P are obtained as the sets of values of the Radon–Nikodym derivatives of these equivalence relations (Theorem 2.1). In §3 we consider the information cocycle $\mathcal{J}_P$ of the shift T_P, defined on the equivalence relation R_P, and prove that the cohomology class of the cocycle $\mathcal{J}_P$ is an invariant of hyperbolic structure preserving isomorphisms (Theorem 3.2). This fact allows us to derive a number of further invariants of the hyperbolic structure of the Markov shift (Remark 3.3). In §4 we turn to finitary isomorphisms and prove that hyperbolic structure preserving isomorphisms need not be finitary (Theorem 4.3). The results of the previous sections imply that finitary isomorphisms need not be isomorphisms of the hyperbolic structures of the Markov shifts. Proposition 4.4, however, shows that finitary isomorphisms with finite expected code (and inverse code) lengths do preserve the hyperbolic structure. The invariants Γ_P, Δ_P and $\mathcal{J}_P$ sometimes make it possible to determine the direction in which an isomorphism of Markov shifts must fail to have finite expected code length (Remark 2.2, Remark 3.4 and Theorem 4.5). For the group Δ_P this result had already been obtained by W. Krieger ([7]).

2. The hyperbolic structure of a Markov shift

Let $P = (P(i, i'), 1 \leqq i, i' \leqq k)$ be an irreducible, stochastic matrix and let $p^- = (p^-(i), 1 \leqq i \leqq k)$ be its left eigenvector with eigenvalue 1 and $\Sigma_{1 \leqq i \leqq k}\, p^-(i) =$

1. We denote by X_P the space of sequences $x = (x_n, n \in \mathbf{Z}) \in \{1, \dots, k\}^{\mathbf{Z}}$ with $P(x_n, x_{n+1}) > 0$ for all $n \in \mathbf{Z}$ and write $T_P : X_P \to X_P$ for the shift $(T_P x)_n = x_{n+1}$. The set X_P is a closed, T_P-invariant subspace of the compact space $\{1, \dots, k\}^{\mathbf{Z}}$, and we define a T_P-invariant Markov probability measure m_P on X_P by setting, for every cylinder set $C = [i_0, \dots, i_s]_r = \{x \in X_P : x_{r+m} = i_m \text{ for } 0 \leqq m \leqq s\}$,

$$m_P(C) = p^{-}(i_0) P(i_0, i_1) \cdots P(i_{s-1}, i_s). \tag{2.1}$$

The automorphism T_P of the probability space (X_P, m_P) is called an *ergodic Markov shift.* For every point $x \in X_P$ we define the *stable* and *unstable manifolds* W_x^s and W_x^u of x by

$$W_x^s = \{x' \in X_P : x'_n = x_n \text{ for all sufficiently large } n > 0\} \tag{2.2}$$

and

$$W_x^u = \{x' \in X_P : x'_{-n} = x_{-n} \text{ for all sufficiently large } n > 0\}. \tag{2.3}$$

If $Q = (Q(j, j'), 1 \leqq j, j' \leqq 1)$ is another irreducible, stochastic matrix and T_Q the associated Markov shift on the shift space (X_Q, m_Q) then we call the shifts T_P and T_Q *isomorphic* if there exists a measure preserving isomorphism $\varphi : X_P \to X_Q$ with

$$\varphi T_P = T_Q \varphi \qquad m_P\text{-a.e.} \tag{2.4}$$

The isomorphism φ is said to be *hyperbolic structure preserving* if there exist null sets $E_P \subset X_P$ and $E_Q \subset X_Q$ such that

$$\varphi(W_x^u \setminus E_P) = W_{\varphi(x)}^u \setminus E_Q \quad \text{and} \quad \varphi(W_x^s \setminus E_P) = W_{\varphi(x)}^s \setminus E_Q \tag{2.5}$$

for every $x \in X_P$. Speaking loosely, a hyperbolic structure preserving isomorphism preserves stable and unstable manifolds up to null sets.

We define an equivalence relation $R_P \subset X_P \times X_P$ on the space X_P by saying that $(x, x') \in R_P$, $x = (x_n)$, $x' = (x'_n)$, if there exist natural numbers m, m', n, n' with

$$x_{-m-s} = x'_{-m'-s} \quad \text{and} \quad x_{n+s} = x'_{n'+s} \tag{2.6}$$

for all $s \geqq 0$. We also define a subrelation R'_P of R_P consisting of all $(x, x') \in R_P$ for which $m = m'$ and $n = n'$ in (2.6). The countable equivalence relations R_P and R'_P are easily seen to be measurable and nonsingular (cf. [4]), and we remark in passing that they are both amenable in the sense of [2]. If R_Q and R'_Q are the corresponding objects for the shift space X_Q, and if $\varphi : X_P \to X_Q$ is a hyperbolic structure preserving isomorphism, then we may — and shall — assume without loss in generality that the null sets E_P and E_Q in (2.4) are saturated with respect to R_P and R_Q, respectively (i.e. unions of equivalence classes). Under this assumption the isomorphism φ satisfies that

$$\varphi^{(2)}(R_P \setminus (E_P \times E_P)) = R_Q \setminus (E_Q \times E_Q) \tag{2.7}$$

and

$$\varphi^{(2)}(R'_P \setminus (E_P \times E_P)) = R'_Q \setminus (E_Q \times E_Q), \tag{2.8}$$

where $\varphi^{(2)}: X_P \times X_P \to X_Q \times X_Q$ is the map $\varphi^{(2)}(x, x') = (\varphi(x), \varphi(x'))$. The Radon–Nikodym derivative r_P of the equivalence relation R_P can be computed explicitly and is given by

$$r_P(x, x') = dm_P(m)/dm_P(x') = \prod_{-m \leqq s \leqq n-1} P(x_s, x_{s+1}) \Big/ \prod_{-m' \leqq s \leqq n'-1} P(x'_s, x'_{s+1}) \tag{2.9}$$

for all $(x, x') \in R_P$ given by (2.6). Since φ is measure preserving, we may assume that

$$r_P(x, x') = r_Q(\varphi(x), \varphi(x')) \tag{2.10}$$

for every $(x, x') \in R_P \setminus (E_P \times E_P)$ (cf. (2.7)). As in [10] we denote by Γ_P the multiplicative group of positive real numbers consisting of all ratios of the form

$$\prod_{0 \leqq s \leqq m} P(i_s, i_{s+1}) \Big/ \prod_{0 \leqq s \leqq n} P(i'_s, i'_{s+1}) \tag{2.11}$$

with $m, n > 0$, $1 \leqq i_s, i'_s \leqq k$, $i_0 = i_{m+1} = i'_0 = i'_{n+1}$, and such that both the numerator and the denominator in (2.11) are nonzero. The subgroup of Γ_P consisting of all ratios of the form (2.11) with $m = n$ will be denoted by Δ_P. Note that $r_P(x, x') \in \Gamma_P$ for all $(x, x') \in R_P$, and that $r_P(x, x') \in \Delta_P$ whenever $(x, x') \in R'_P$. We also recall from [10] that the quotient group Γ_P / Δ_P is cyclic: there exists a positive real number c_P with

$$\prod_{0 \leqq s \leqq m} P(i_s, i_{s+1}) \in c_P^m \Delta_P \tag{2.12}$$

whenever $m > 0$, $1 \leqq i_s \leqq k$, $i_0 = i_m$, and $P(i_0, i_1) \cdots P(i_m, i_{m+1}) > 0$. By combining (2.5) with (2.7)–(2.12) we have proved the following theorem.

2.1. Theorem. *Let P and Q be irreducible, stochastic matrices and assume that there exists a hyperbolic structure preserving isomorphism $\varphi: X_P \to X_Q$ between the corresponding Markov shifts T_P on (X_P, m_P) and T_Q on (X_Q, m_Q). Then $\Gamma_P = \Gamma_Q$, $\Delta_P = \Delta_Q$ and $c_P^d \Delta_P = c_Q^d \Delta_Q$, where d denotes the period of P (or Q).*

2.2. Remark. Let P and Q be irreducible, stochastic matrices and assume that there exists a measure preserving isomorphism $\varphi: X_P \to X_Q$ of the Markov shifts T_P and T_Q and a null set $E_P \subset X_P$ with the property that $\varphi(W^u_x \setminus E_P) \subset W^u_{\varphi(x)}$ and $\varphi(W^s_x \setminus E_P) \subset W^s_{\varphi(x)}$ for every $x \in X_P$. From (2.7) and (2.8) it is clear that $\varphi^{(2)}(R_P \setminus (E_P \times E_P)) \subset R_Q$ and $\varphi^{(2)}(R'_P \setminus (E_P \times E_P)) \subset R'_Q$, and (2.9)–(2.12) imply that, under these assumptions, $\Gamma_P \subset \Gamma_Q$, $\Delta_P \subset \Delta_Q$ and $c^d_P \Delta_P \subset c^d_Q \Delta_Q$.

3. The information cocycle

Let $P = (P(i, i'), 1 \leqq i, i' \leqq k)$ be an irreducible, stochastic matrix, and let R_P denote the equivalence relation introduced in (2.6). We define the information cocycle $\mathcal{J}_P(\cdot, \cdot): R_P \to \mathbf{R}$ of the Markov shift T_P (or, more precisely, of the equivalence relation R_P) by setting

$$\mathcal{J}_P(x, x') = \log \left\{ \prod_{0 \leqq s \leqq n-1} P(x_s, x_{s+1}) \Big/ \prod_{0 \leqq s \leqq n'-1} P(x'_s, x'_{s+1}) \right\} \tag{3.1}$$

whenever $(x, x') \in R_P$ are given as in (2.6) (for the general terminology we refer to [4], [8] and [11]). This definition is — apart from an inessential generalization — equivalent to the one given in [1] and [12] and can be put into a more familiar form by introducing the group $[R_P]$ of all nonsingular automorphisms V of (X_P, m_P) with $(Vx, x) \in R_P$ for m_P-a.e. $x \in X_P$, and by setting

$$J_P(V, x) = \mathcal{J}_P(Vx, x) \tag{3.2}$$

for every $V \in [R_P]$, $x \in X_P$. The map $J_P : [R_P] \times X_P \to \mathbf{R}$ satisfies the cocycle equation

$$J_P(VW, x) = J_P(V, Wx) + J_P(W, x) \qquad m_P\text{-a.e.} \tag{3.3}$$

for all $V, W \in [R_P]$. For $V = T_P$, $J_P(T_P, \cdot)$ is essentially the information function of the Markov shift

$$J_P(T_P, x) = -\log P(x_0, x_1) \tag{3.4}$$

for every $x = (x_n) \in X_P$. In particular,

$$\int J_P(T_P, \cdot)\, dm_P = h(T_P), \tag{3.5}$$

where $h(T_P)$ is the entropy of T_P. We denote by Γ'_P and Δ'_P the (additive) subgroups of $\mathbf{R}$ given by

$$\Gamma'_P = \{\log \gamma : \gamma \in \Gamma_P\}, \qquad \Delta'_P = \{\log \delta : \delta \in \Delta_P\}. \tag{3.6}$$

Remark 5.3 in [10] implies the existence of a function $u: X_P \to \mathbf{R}$ depending only on the zero coordinate (i.e. $u(x) = u(x_0)$) such that

$$\mathcal{J}_P(x, x') - u(x) + u(x') \in \Gamma'_P \tag{3.7}$$

for every $(x, x') \in R_P$. This allows us to define cocycles $\mathcal{J}'_P$: $R_P \to \Gamma'_P$ and J'_P: $[R_P] \to \Gamma'_P$ by setting

$$\mathcal{J}'_P(x, x') = \mathcal{J}_P(x, x') - u(x) + u(x') \tag{3.8}$$

and

$$J'_P(V, x) = J_P(V, x) - u(Vx) + u(x) \tag{3.9}$$

for every $(x, x') \in R_P$ and $V \in [R_P]$.

Now assume that $Q = (Q(j, j'), 1 \leqq j, j' \leqq 1)$ is another irreducible, stochastic matrix and that there exists a hyperbolic structure preserving isomorphism $\varphi: X_P \to X_Q$ of the corresponding Markov shifts T_P and T_Q. We write Γ'_Q, Δ'_Q for the groups defined by (3.6) with Q replacing P and denote by $\mathcal{J}_Q$, J_Q, $\mathcal{J}'_Q$ and J'_Q the four versions of the information cocycle for the Markov shift T_Q analogous to (3.1), (3.2), (3.8) and (3.9). According to (2.7) we can define a cocycle $\varphi_* \mathcal{J}'_Q$: $R_P \to \Gamma'_P = \Gamma'_Q$ by setting

$$\varphi_* \mathcal{J}'_Q(x, x') = \mathcal{J}'_Q(\varphi(x), \varphi(x')) \tag{3.10}$$

for all $(x, x') \in R_P \setminus (E_P \times E_P)$, and by setting $\varphi_* \mathcal{J}'_Q = 0$ on $R_P \cap (E_P \times E_P)$. By $\varphi_* J'_Q$: $[R_P] \times X_P \to \Gamma'_P$ we denote the cocycle

$$\varphi_* J'_Q(V, x) = \varphi_* \mathcal{J}'_Q(Vx, x). \tag{3.11}$$

3.1. Theorem. *Let P and Q be irreducible, stochastic matrices and assume that there exists a hyperbolic structure preserving isomorphism $\varphi: X_P \to X_Q$ of the corresponding Markov shifts T_P and T_Q. Then the cocycles $\mathcal{J}'_P$ and $\varphi_* \mathcal{J}'_Q$ (or, equivalently, J'_P and $\varphi_* J'_Q$) are cohomologous. In other words, there exists a measurable function $f: X_P \to \Gamma'_P$ such that, for every $V \in [R_P]$,*

$$J'_P(V, x) - \varphi_* J'_Q(V, x) = f(Vx) - f(x) \qquad m_P\text{-a.e.} \tag{3.12}$$

Proof. We shall prove (3.12) by establishing it for all V in successively bigger subgroups of $[R_P]$, but first we have to set up some notation. Denote by $\alpha_P = \{[1]_0, \ldots, [k]_0\}$ the state partition of X_P and let, for every r, s with $-\infty \leqq r < s \leqq \infty$, $\mathcal{A}_P(r, s)$ be the σ-algebra generated by $\{T_P^{-m} \alpha_P : r \leqq m \leqq s\}$. For every $n \in \mathbf{Z}$ and $x \in X_P$ we define sets $W^s_x(P, n)$ and $W^u_x(P, n)$ by

(3.13) $$W_x^s(P,n)=\{x'\in X_P\colon x'_m=x_m \text{ for all } m\geqq n\}$$

and

(3.14) $$W_x^u(P,n)=\{x'\in X_P\colon x'_m=x_m \text{ for all } m\leqq n\}.$$

For fixed n, the sets $\{W_x^s(P,n)\colon x\in X_P\}$ form the atoms of the σ-algebra $\mathcal{A}_P(n,\infty)$, and we write $\{\mu_x^{(P,n)}\colon x\in X_P\}$ for the associated decomposition of the measure m_P with the properties that $\mu_x^{(P,n)}(W_x^s(P,n))=1$ for all x and that $\mu_x^{(P,n)}(B)=E(\chi_B \mid \mathcal{A}_P(n,\infty))(x)$ m_P-a.e. for all Borel sets $B\subset X_P$, where χ_B is the indicator function of B, and where $E(\cdot\mid\cdot)$ denotes conditional expectation with respect to the measure m_P. The analogous decomposition of m_P with respect to the σ-algebra $\mathcal{A}_P(-\infty,n)$ will be denoted by $\{\nu_x^{(P,n)}\colon x\in X_P\}$. By α_Q, $\mathcal{A}_Q(r,s)$, $W_y^s(Q,n)$, $W_y^u(Q,n)$, $\mu_y^{(Q,n)}$ and $\nu_y^{(Q,n)}$ we denote the analogous objects for the space X_Q. Our assumptions on φ imply that, for m_P-a.e. $x\in X_P$, and for every $n\in\mathbf{Z}$,

$$\mu_x^{(P,n)}\left\{\bigcup_{m\in\mathbf{Z}}\varphi^{-1}(W^s_{\varphi(x)}(Q,m))\right\}=1$$

and

$$\nu_x^{(P,n)}\left\{\bigcup_{m\in\mathbf{Z}}\varphi^{-1}(W^u_{\varphi(x)}(Q,m))\right\}=1.$$

For every $\varepsilon>0$ we can thus find an integer $M=M(\varepsilon)>0$ such that, for every $n\in\mathbf{Z}$, the sets

(3.15) $$D^s(n,\varepsilon)=\{x\in X_P\colon \mu_x^{(P,n)}(\varphi^{-1}(W^s_{\varphi(x)}(Q,n+M)))>1-\varepsilon\}$$

and

(3.16) $$D^u(n,\varepsilon)=\{x\in X_P\colon \nu_x^{(P,n)}(\varphi^{-1}(W^u_{\varphi(x)}(Q,n-M)))>1-\varepsilon\}$$

satisfy that

(3.17) $$m_P(D^s(n,\varepsilon))>1-\varepsilon$$

and

(3.18) $$m_P(D^u(n,\varepsilon))>1-\varepsilon.$$

With this notation at hand we turn to the proof of (3.12). Let

$$F_P^+(n)=\{V\in[R_P]\colon (Vx)_m=x_m\ m_P\text{-a.e., for every } m\leqq n\},$$

$$F_P^-(n)=\{V\in[R_P]\colon (Vx)_m=x_m\ m_P\text{-a.e., for every } m\geqq n\},$$

and let

$$F_P^+ = F_P^+(0), \qquad F_P^- = F_P^-(0).$$

By F_0 we denote the subgroup of all measure preserving elements in $[R_P]$. Fix $\varepsilon > 0$ and choose $M = M(\varepsilon)$ as in (3.15)–(3.18). Clearly, $J'_P(V, \cdot) = 0$ for all $V \in F_P^+ \cap F_0$. We apply (3.16) and (3.18) with $n = M$ to conclude that, for every measure preserving $V \in F_P^+(M)$, and for m_P-a.e. $x \in D''(M, \varepsilon)$,

$$\mu_x^{(P,M)}(\{x' : (\varphi(x'))_m = (\varphi(Vx'))_m \text{ for all } m \leqq 0\}) > 1 - 2\varepsilon.$$

From the fact that $\varphi V \varphi^{-1}$ preserves m_Q and from the definition of $\varphi_* J'_Q$ (cf. (3.1), (3.9) and (3.11)) it is clear that, for a.e. $x \in D''(M, \varepsilon)$,

$$\mu_x^{(P,M)}(\{x' : \varphi_* J'_Q(V, x') \neq 0\}) < 2\varepsilon, \tag{3.19}$$

so that

$$m_P(\{x' : \varphi_* J'_Q(V, x') \neq 0\}) < 3\varepsilon \tag{3.20}$$

for every $V \in F_P^+(M) \cap F_0$. Every $W \in F_P^+$ can be written as $W = T_P^M V T_P^{-M}$ with $V \in F_P^+(M)$, so that

$$\varphi_* J'_Q(W, x) = \varphi_* J'_Q(T_P^M, V T_P^{-M} x) - \varphi_* J'_Q(T_P^M, T_P^{-M} x) + \varphi_* J'_Q(V, T_P^{-M} x).$$

By varying ε we conclude from (3.20) that the cocycle J'_Q is bounded on $F_P^+ \cap F_0$ in the following sense: for every $\delta > 0$ there exists a finite set $C \subset \Gamma'_P$ such that $m_P(\{x : \varphi_* J'_Q(W, x) \notin C\}) < \delta$ for all $W \in F_P^+ \cap F_0$. By [8] or [11], the cocycle $\varphi_* J'_Q$ is a coboundary on $F_P^+ \cap F_0$, i.e. there exists a measurable function $g : X_P \to \Gamma'_P$ with $\varphi_* J'_Q(W, x) = g(Wx) - g(x)$ m_P-a.e., for every $W \in F_P^+ \cap F_0$. Our next aim is to prove that

$$\varphi_* J'_Q(W, x) = \log((dm_P W / dm_P)(x)) + g(Wx) - g(x) \qquad m_P\text{-a.e.}, \tag{3.21}$$

for every $W \in F_P^+$ (note that $\log((dm_P W / dm_P)(x)) = \log r_P(Wx, x) \in \Gamma'_P$ a.e. — cf. (2.9)). We fix $W \in F_P^+$. An elementary argument shows that there exists a sequence $(W_n : n \geqq 1) \subset F_p^+$ with $(W_n x)_m = x_m$ m_P-a.e., for all $m \leqq n$, and such that WW_n preserves m_P for all $n \geqq 1$. Clearly, $\lim_n f \cdot W_n = f$ in the topology of convergence in measure, for every measurable function $f : X_P \to \Gamma'_P$. The definition of $\varphi_* J'_Q(W_n, \cdot)$, together with (3.16) and (3.18), implies that $\lim_n (\varphi_* J'_Q(W_n, \cdot) - \log(dm_P W_n / dm_P)) = 0$ in the topology of convergence in measure. Furthermore,

$$\log(dm_P WW'_n / dm_P) = \log(dm_P W / dm_P) \cdot W'_n + \log(dm_P W'_n / dm_P) = 0 \text{ for every } n,$$

so that

$$g\cdot W-g=\lim_n (g\cdot WW'_n-g)$$
$$=\lim_n \varphi_* J'_Q(WW'_n,\cdot)$$
$$=\varphi_* J'_Q(W,\cdot)-\log(dm_P W/dm_P).$$

This proves (3.21) for $W\in F_P^+$. Since $J'_P(W,\cdot)=\log(dm_P W/dm_P)$ for all $W\in F_P^+$, we have proved that the cocycle $c=J'_P-\varphi_* J'_Q$ has the form

$$c(W,x)=g(Wx)-g(x)\qquad m_P\text{-a.e.,}\tag{3.22}$$

for every $W\in F_P^+$. In a similar manner one can apply (3.15)–(3.18) to show that, for every measure preserving $V\in F_P^-(-M)$, and for a.e. $x\in D^s(-M,\varepsilon)$,

$$\nu_x^{(P,-M)}(\{x'\colon \varphi_* J'_Q(V,x')\neq 0\})<2\varepsilon^2,\tag{3.23}$$

so that

$$m_P(\{x'\colon \varphi_* J'_Q(V,x')\neq 0\})<3\varepsilon^2\tag{3.24}$$

for all such V. As before we conclude the existence of a measurable function $h\colon X_P\to\Gamma'_P$ with

$$c(W,x)=h(Wx)-h(x)\qquad m_P\text{-a.e.,}\tag{3.25}$$

for every $W\in F_P^-$. In order to combine (3.22) and (3.25) we write F^* for the group $F_P^-\cdot F_P^+=\{VW\colon V\in F_P^-,\ W\in F_P^+\}$. Since

$$c(VWx,x)=h(VWx)-h(Wx)+g(Wx)-g(x)\quad \text{for every } V\in F_P^-,\ W\in F_P^+,$$

the cocycle c is bounded on the group $F^*\cap F_0$ in the sense just described, and we conclude the existence of a single measurable function $f'\colon X_P\to\Gamma'_P$ with

$$c(V,x)=f'(Vx)-f'(x)\qquad m_P\text{-a.e.,}\tag{3.26}$$

for every $V\in F^*$. We continue by setting, for every $V\in[R_P]$ and $x\in X_P$,

$$c'(V,x)=c(V,x)-f'(Vx)+f'(x).$$

From (3.26) it follows in particular that $c'(T_P^nVT_P^{-n},T_P^nx)=c'(T_P^n,Vx)-c'(T_P^n,x)=0$ a.e., whenever $V\in F^*$ and $T_P^nVT_P^{-n}\in F^*$. For fixed n, the set $\{V\in F^*\colon T_P^nVT_P^{-n}\in F^*\}$ is a subgroup of F^* whose ergodic components are given by $[i]_0\cap[i']_n$, $1\leqq i,i'\leqq k$. For every $n\geqq 1$ the function $c'(T_P^n,\cdot)$ thus only depends on the coordinates 0 and n, and we conclude that $c'(T_P,x)=\alpha+u(T_Px)-u(x)$ a.e., where $\alpha\in\mathbf{R}$ and where $u\colon X_P\to\mathbf{R}$ is a function of the zero coordinate: $u(x) = u(x_0)$. Finally we note that

$$\alpha = \int c(T_P, \cdot) dm_P = \int J'_P(T_P, \cdot) dm_P - \int J'_Q(T_Q, \cdot) dm_Q = h(T_P) - h(T_Q) = 0.$$

By adding a constant to u in order to make sure that $u(x) \in \Gamma'_P$ for all x and by setting $f = f' + u$ we have proved (3.12) for all V in the group G generated by T_P and F^*. We denote by $[G]$ the group $\{V \in [R_P]: Vx \in Gx\ m_P\text{-a.e.}\}$, observe that $[G] = [R_P]$, and note that (3.12) holds for all $V \in [G]$. This completes the proof of the theorem. □

3.2. Remark. Theorem 3.1 implies in particular that the information cocycles $\mathcal{J}_P$ and $\varphi_* \mathcal{J}_Q$ are cohomologous. Hence the information functions I_P and I_Q of the Markov shifts T_P and T_Q are related by the equation

$$(3.27)\quad I_P(x) = \log(p^-(x_1)/p^-(x_0)P(x_0, x_1)) = I_Q(\varphi(x)) + g(T_Px) - g(x) \quad m_P\text{-a.e.},$$

where $g: X_P \to \mathbf{R}$ is a measurable function. Furthermore, if $\eta \in \mathrm{Hom}(\Gamma'_P, \mathbf{R}) = \mathrm{Hom}(\Gamma'_Q, \mathbf{R})$, Theorem 3.1 implies that the numbers $h_{P,\eta}$ and $\sigma^2_{P,\eta}$ described in [10, Theorems 6.1 and 6.2] satisfy that $h_{P,\eta} = h_{Q,\eta}$ and $\sigma^2_{P,\eta} = \sigma^2_{Q,\eta}$. The set $\{(h_{P,\eta}, \sigma^2_{P,\eta}): \eta \in \mathrm{Hom}(\Gamma'_P, \mathbf{R})\}$ is thus an invariant of the hyperbolic structure.

3.3. Remark. We denote by $\mathcal{A}_P = \mathcal{A}_P(0, \infty)$ and $\mathcal{A}_Q = \mathcal{A}_Q(0, \infty)$ the past σ-algebras of the shifts T_P and T_Q, respectively. Formulae (3.15)–(3.18) and their analogues for φ^{-1} can be used to show that the information functions $I(\mathcal{A}_P \mid \varphi^{-1}(\mathcal{A}_Q))$ and $I(\varphi^{-1}(\mathcal{A}_Q) \mid \mathcal{A}_P)$ are both finite a.e., so that the σ-algebras $\mathcal{A}_P$ and $\varphi^{-1}(\mathcal{A}_Q)$ are I-related in the sense of [1, Definition 2.1]. Theorem 3.1 is then a consequence of [1, Theorem 3.5], and the function f in (3.12) differs from $I(\varphi^{-1}(\mathcal{A}_Q) \mid \mathcal{A}_P) - I(\mathcal{A}_P \mid \varphi^{-1}(\mathcal{A}_Q))$ by a function depending only on the zero coordinate. For further details we refer to [1].

3.4. Remark. A closer look at the proof of Theorem 3.1 reveals that we have not made full use of the fact that φ preserves the hyperbolic structure of the shift spaces and that we have proved the following more general result. Let P and Q be irreducible, stochastic matrices and assume that there exists an isomorphism $\varphi: X_P \to X_Q$ of the Markov shifts T_P and T_Q and a null set $E_P \subset X_P$ such that $\varphi(W^u_x \setminus E_P) \subset W^u_{\varphi(x)}$ and $\varphi(W^s_x \setminus E_Q) \subset W^s_{Q(x)}$ for every $x \in X_P$. Then there exists a measurable function $f: X_P \to \Gamma'_P$ such that, for every $V \in [R_P]$,

$$(3.28)\qquad J'_P(V, x) - \varphi_* J'(V, x) = f(Vx) - f(x) \qquad m_P\text{-a.e.}$$

(cf. Remark 2.2). This "asymmetric" version of Theorem 3.1 does not appear to be an immediate consequence of the methods described in Remark 3.3.

4. Finitary isomorphisms and the hyperbolic structure

Let $P = (P(i, i'),\ 1 \leqq i, i' \leqq k)$ and $Q = (Q(j, j'),\ 1 \leqq j, j' \leqq l)$ be irreducible, stochastic matrices and let T_P and T_Q be the associated Markov shifts on the shift spaces (X_P, m_P) and (X_Q, m_Q), respectively. An isomorphism $\varphi: X_P \to X_Q$ of the two shifts is called *finitary* if there exist measurable functions $a_\varphi, m_\varphi : X_P \to \mathbf{N}$, $a_{\varphi^{-1}}, m_{\varphi^{-1}}: X_Q \to \mathbf{N}$ and (shift invariant) null sets $N_P \subset X_P$, $N_Q \subset X_Q$ such that

$$\varphi(x)_0 = \varphi(x')_0 \tag{4.1}$$

for all $x, x' \in X_P \setminus N_P$ with $x_n = x'_n$ for $-m_\varphi(x) \leqq n \leqq a_\varphi(x)$, and

$$\varphi^{-1}(y)_0 = \varphi^{-1}(y')_0 \tag{4.2}$$

for all $y, y' \in X_Q \setminus N_Q$ with $y_n = y'_n$ for $-m_{\varphi^{-1}}(y) \leqq n \leqq a_{\varphi^{-1}}(y)$. In this section we shall prove that hyperbolic structure preserving isomorphisms are — in general — not finitary.

In order to construct an example of such an isomorphism we set $X = \{1, -1\}^{\mathbf{Z}}$, $\mu = \nu^{\mathbf{Z}}$ with $\nu(1) = \nu(-1) = \frac{1}{2}$, and denote by T the shift $(Tx)_n = x_{n+1}$ on (X, μ). For every $n \geqq 1$ and every $(i_{-n}, \ldots, i_n) \in \{1, -1\}^{2n+1} = X(n)$, we choose a cylinder set $C(i_{-n}, \ldots, i_n) = [j_{-n!}, \ldots, j_{n!}]_{-n!} \subset X$ with $j_k = i_k$ for $-n \leqq k \leqq n$. Put $B_n = \bigcup_{X(n)} C(i_{-n}, \ldots, i_n)$ and $A_n = B_1 \triangle B_2 \triangle \cdots \triangle B_n$. The set $A = \lim_n \sup A_n$ satisfies that $\mu(\mathcal{O} \cap A) > 0$ and $\mu(\mathcal{O} \setminus A) > 0$ for every open set $\mathcal{O} \subset X$. The indicator function of the set A will be denoted by χ_A.

We recall the definition of the equivalence relation R' from Section 2 (cf. (2.6)), write $[R']$ for the group of (necessarily measure preserving) nonsingular automorphisms V of (X, μ) satisfying that $(Vx, x) \in R'$ for μ-a.e. $x \in X$ and put, for every $M \in \mathbf{Z}$,

$$G^+(M) = \{V \in [R'] \colon (Vx)_m = x_m \ \mu\text{-a.e., for every } m \leqq M\}$$

and

$$G^-(M) = \{V \in [R'] \colon (Vx)_m = x_m \ \mu\text{-a.e., for every } m \geqq M\}.$$

4.1. Lemma. *For every $\gamma \in G^- = G^-(0)$, the sum*

$$d^-(\gamma, \cdot) = \sum_{n \geqq 0} 3^n (\chi_A T^n \gamma - \chi_A T^n) \tag{4.3}$$

converges in measure, and the resulting cocycle $d^-: G^- \times X \to \mathbf{Z}$ is a coboundary, i.e. there exists a measurable function $f^-: X' \to \mathbf{Z}$ with $d^-(\gamma, x) = f^-(\gamma x) - f^-(x)$ μ-a.e., for every $\gamma \in G^-$.

PROOF. For every $\gamma \in G^-(M)$ with $M \leqq 0$ and every $n \geqq 0$ we have that

$$\mu(T^{-n}A \,\triangle\, \gamma T^{-n}A) \leqq 2 \sum_{\{k:k!>n-M\}} \mu(B_k) = 2 \sum_{\{k:k!>n-M\}} 2^{2k-2k!}.$$

Hence

$$\mu(\{x: d^-(\gamma, x) \neq 0\} \leqq \sum_{n \geqq 0} \mu(T^{-n}A \,\triangle\, \gamma T^{-n}A) \leqq 2 \sum_{\{k:k!>-M\}} k! \cdot 2^{2k-2k!}.$$

Clearly

$$\sum_{\{k:k!>-M\}} k! \cdot 2^{2k-2k!} < \infty,$$

and

$$\lim_{M \to -\infty} \sum_{\{k:k!>-M\}} k! \cdot 2^{2k-2k!} = 0.$$

From these estimates it follows that the cocycle d^-: $G^- \times X \to \mathbf{Z}$ is not only well defined, but that it is also bounded on G^- in the sense described in the proof of Theorem 3.2 (cf. [8] and [11]). Hence d^- is a coboundary. □

In exactly the same manner one can prove the next lemma.

4.2. LEMMA. *For every $\gamma \in G^+ = G^+(0)$, the sum*

$$d^+(\gamma, \cdot) = \sum_{n \geqq 0} 3^n (\chi_A T^{-n}\gamma - \chi_A T^{-n}) \tag{4.4}$$

converges in measure, and the resulting cocycle d^+: $G^+ \times X \to \mathbf{Z}$ is a coboundary.

With these results at hand we can now describe an example of a hyperbolic structure preserving isomorphism which is not finitary. Let $T_P = T_Q = T \times T$, acting on $(X_P, m_P) = (X_Q, m_Q) = (X \times X, \mu \times \mu)$ (this is just the four-shift on symbols of equal probability, written in a specific way). A typical element $x \in X_P$ will be written in the form

$$x = (u, v) = ((u_n, v_n), n \in \mathbf{Z}), \tag{4.5}$$

where $u = (u_n)$ and $v = (v_n)$ are elements of X. We define a measure preserving isomorphism φ: $X_P \to X_Q$ by setting, for every $n \in \mathbf{Z}$ and every $x = ((u_n, v_n), n \in \mathbf{Z}) \in X_P$,

$$\varphi(x)_n = (u_n, v_n) \text{ if } v \notin T^{-n}A \text{ and } \varphi(x)_n = (-u_n, v_n) \text{ if } v \in T^{-n}A. \tag{4.6}$$

From the definition of φ it is obvious that $\varphi T_P = T_Q \varphi$.

4.3. THEOREM. *The isomorphism $\varphi: X_P \to X_Q$ defined by (4.6) is hyperbolic structure preserving, but not finitary.*

PROOF. First we prove that φ is hyperbolic structure preserving. Choose a measurable function $f^-: X \to \mathbf{Z}$ as in Lemma 4.1 such that $f^-\gamma - f^- = d^-(\gamma, \cdot)$ μ-a.e., for every $\gamma \in G^-$. For every $m \in \mathbf{Z}$, let $D'_-(m) = \{x \in X: f^-(x) = m\}$ and $D_-(m) = X \times D'_-(m) \subset X_P$. We fix $m \in \mathbf{Z}$ for the moment and claim that, for m_P-a.e. $x \in D_-(m)$, and for $\mu_x^{(P,0)}$-a.e., $x' \in D_-(m)$,

$$\varphi(x)_n = \varphi(x')_n \qquad \text{for all } n \geqq 0. \tag{4.7}$$

Indeed, if (4.7) is violated, we can find an integer $k \geqq 0$ and two disjoint subsets F_1, F_2 of $D_-(m)$ with the following properties (for notation we refer to (3.13)):

(4.8) the set $F = \{x: \mu_x^{(P,0)}(F_i) > 0 \text{ for } i = 1, 2\}$ satisfies that $m_P(F) > 0$,

and

(4.9) $\varphi(x')_k \neq \varphi(x'')_k$ for all $x \in F$, $x' \in W_x^s(P,0) \cap F_1$ and $x'' \in W_x^s(P,0) \cap F_2$.

Condition (4.9) implies in particular that $v' \in T^{-k}A$ and $v'' \notin T^{-k}A$ (or vice versa), where $x' = (u', v')$ and $x'' = (u'', v'')$ (cf. (4.5)), since $x'_k = x''_k$ (cf. (4.6)). The action of G^- is ergodic with respect to all the measures $\mu_x^{(P,0)}$, $x \in X_P$. As a consequence of (4.8) we can find a $\gamma \in G^-$ with $m_P(\gamma F_1 \cap F_2) > 0$, and (4.3) implies that $d^-(\gamma, x) \neq 0$ for m_P-a.e. $x \in F_1 \cap \gamma^{-1}F_2$. Since both $F_1 \cap \gamma^{-1}F_2$ and $\gamma(F_1 \cap \gamma^{-1}F_2) = \gamma F_1 \cap F_2$ are subsets of positive measure in $D_-(m)$, we obtain a contradiction to Lemma 4.1 and to our choice of $D_-(m)$. This proves (4.7). Now we can find, for a.e. $x \in D_-(m)$, and for every $\varepsilon > 0$, an integer $N(x) \leqq 0$ such that (4.7) is satisfied for all $x' \in W_x^s(P, N(x))$ with the exception of a set of $\mu_x^{(P,N(x))}$-measure $\leqq \varepsilon$. By varying m and by using the fact that $\varphi T_P = T_Q \varphi$ we conclude the existence of an m_P-null set $E_1 \subset X_P$ with $\varphi(W_x^s \setminus E_1) \subset W_{\varphi(x)}^s$ for all $x \in X_P$. A similar proof, using Lemma 4.2, yields the existence of a null set $E_2 \subset X_P$ with $\varphi(W_x^u \setminus E_2) \subset W_{\varphi(x)}^u$ for all $x \in X_P$. The situation is obviously symmetric in P and Q, and we conclude that φ satisfies (2.5), i.e. that φ is hyperbolic structure preserving.

In order to see that φ is not finitary we recall the fact that, for every open set $\mathcal{O} \subset X$, $\mu(A \cap \mathcal{O}) > 0$ and $\mu(\mathcal{O} \setminus A) > 0$. From (4.6) it is now clear that there cannot exist a nonempty, open set in X_P on which the map $x \to \varphi(x)_0$ from X_P to $\{1, -1\}$ is constant a.e. This completes the proof of Theorem 4.3. □

The invariants constructed in this paper imply that finitary isomorphisms need not preserve the hyperbolic structure, and Theorem 4.3 shows that the hyperbolic structure may be preserved under maps which are not finitary. There exists, however, a very interesting class of isomorphisms which are both finitary and hyperbolic structure preserving. Recall that a finitary isomorphism $\varphi: X_P \to X_Q$ of two Markov shifts T_P and T_Q is said to have *finite expected code length* if the functions a_φ and m_φ in (4.1) are integrable. Following [7] we note that, in this case, the functions

$$a^*_\varphi(x) = \sup_{n \geqq 0} (a_\varphi(T_P^{-n}x) - n) \tag{4.10}$$

and

$$m^*_\varphi(x) = \sup_{n \geqq 0} (m_\varphi(T_P^{n}x) - n) \tag{4.11}$$

are both finite a.e., From these definitions it is clear that, for every x with $a^*_\varphi(x) < \infty$,

$$\varphi(W^u_x(P, a^*_\varphi(x)) \setminus N_P)) \subset W^u_{\varphi(x)}(Q, 0). \tag{4.12}$$

Similarly, if $m^*_\varphi(x) < \infty$, then

$$\varphi(W^s_x(P, -m^*_\varphi(x)) \setminus N_P)) \subset W^s_{\varphi(x)}(Q, 0). \tag{4.13}$$

4.4. PROPOSITION. *Let $\varphi: X_P \to X_Q$ be a finitary isomorphism of the Markov shifts T_P and T_Q with finite expected code length or, more generally, a finitary isomorphism with $a^*_\varphi < \infty$ and $m^*_\varphi < \infty$ m_P-a.e. Then there exists a null set $E_P \subset X_P$ with*

$$\varphi(W^u_x \setminus E_P) \subset W^u_{\varphi(x)} \quad \text{and} \quad \varphi(W^s_x \setminus E_P) \subset W^s_{\varphi(x)} \tag{4.14}$$

for every $x \in X_P$. If both φ and φ^{-1} have finite expected code length, then φ is hyperbolic structure preserving.

PROOF. The proof of this result is implicit in [7]. Choose $M > 0$ such that the set $D = \{x : a^*_\varphi(x) \leqq M\}$ has positive measure. We fix $n \in \mathbf{Z}$ and define, for every $x \in X_P$, the set $W^u_x(P, n)$ by (3.14). For a.e. $x \in X_P$ we can find an integer $m > M - n$ with $x' = T_P^{-m}x \in D$. By (4.12),

$$\varphi(W^u_{x'}(P, n+m) \setminus N_P) \subset \varphi(W^u_{x'}(P, a^*_\varphi(x')) \setminus N_P) \subset W^u_{\varphi(x')}(Q, 0),$$

so that $\varphi(W^u_x(P, n) \setminus N_P) \subset W^u_{\varphi(x)}(Q, -m)$ (we are assuming without loss of generality that $\varphi(T_P x) = T_Q \varphi(x)$ for every $x \in X_P \setminus N_P$). This implies the first half of (4.14), and the second half is proved in exactly the same way. □

4.5. THEOREM. *Let P and Q be irreducible, stochastic matrices and let T_P and T_Q be the corresponding Markov shifts on the shift spaces (X_P, m_P) and (X_Q, m_Q), respectively. Assume that there exists a finitary isomorphism $\varphi: X_P \to X_Q$ of the two shifts with finite expected code length (or with $a^*_\varphi < \infty$ and $m^*_\varphi < \infty$ m_P-a.e.). Then $\Gamma_P \subset \Gamma_Q$, $\Delta_P \subset \Delta_Q$, and $c^d_P\Delta_P \subset c^d_Q\Delta_Q$, where d denotes the period of P (or Q).*

PROOF. This is an immediate consequence of Proposition 4.4 and Remark 2.2. □

4.6. REMARK. An isomorphism $\varphi: X_P \to X_Q$ of the Markov shifts T_P and T_Q is said to satisfy the *hypothesis F.E.* if both φ and φ^{-1} have finite expected code lengths. If no such isomorphism exists between T_P and T_Q, Theorem 4.5 can be useful in determining the direction in which φ has to fail to have finite expected code length. W. Krieger has proved in [7] that, if $\varphi: X_P \to X_Q$ is an isomorphism with finite expected code length, then $\Delta_P \subset \Delta_Q$. The corresponding assertion for the Γ-groups answers a question posed in [10, p. 31]. In this context we also refer to the "asymmetric" versions of the Theorems 2.1 and 3.1 discussed in the Remarks 2.2 and 3.4, respectively.

4.7. PROBLEM. Assume that there exists a hyperbolic structure preserving isomorphism $\varphi: X_P \to X_Q$ of the Markov shifts T_P and T_Q. Does there also exist an isomorphism which satisfies the hypothesis F.E.? A particularly intriguing aspect of this question concerns the β-functions of the Markov shifts (cf. [14], [12]): for every $t \in \mathbf{R}$, the value $\beta_P(t)$ of the β-function of the Markov shift T_P is defined as the maximal eigenvalue of the matrix $(P(i, i')^t, 1 \leqq i, i' \leqq k)$ obtained by raising all nonzero entries of the matrix P to the power t. The main result in [12] sates that $\beta_P \equiv \beta_Q$ whenever there exists an isomorphism of the Markov shifts T_P and T_Q which satisfies the hypothesis F.E. (or, more generally, for which the functions a^*_φ, m^*_φ, $a^*_{\varphi^{-1}}$ and $m^*_{\varphi^{-1}}$ are all finite a.e. — cf. [13]). Does the existence of a hyperbolic structure preserving isomorphism of the Markov shifts also imply that $\beta_P \equiv \beta_Q$ or, equivalently, is the β-function an invariant of the hyperbolic structure? For finitary, hyperbolic structure preserving isomorphisms this question will be answered affirmatively in [13].

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